Computable Integrability.
Chapter 2: Riccati equation

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Contents

1 Introduction

Riccati equation (RE)

$$\boxed{\phi_x = a(x)\phi^2 + b(x)\phi + c(x)} \quad (1)$$

is one of the most simple nonlinear differential equations because it is of **first order** and with **quadratic nonlinearity**. Obviously, this was the reason that as soon as Newton invented differential equations, RE was the first one to be investigated extensively since the end of the 17th century [1]. In 1726 Riccati considered the first order ODE

$$w_x = w^2 + u(x)$$

with polynomial in x function $u(x)$. Evidently, the cases $\deg u = 1, 2$ correspond to the Airy and Hermite transcendent functions, respectively. Below we show that Hermite transcendent is integrable in quadratures. As to Airy transcendent, it is only F-integrable[1] though the corresponding equation itself is at the first glance a simpler one.

Thus, new transcendents were introduced as solutions of the first order ODE with the quadratic nonlinearity, i.e. as solutions of REs. Some classes of REs are known to have general solutions, for instance:

$$y' + ay^2 = bx^\alpha$$

where all a, b, α are constant in respect to x. D. Bernoulli discovered(1724-25) that this RE is integrable in elementary functions if $\alpha = -2$ or $\alpha = -4k(2k-1), k = 1, 2, 3, ...$ Below some general results about RE are presented which make it widely usable for numerous applications in different branches of physics and mathematics.

2 General solution of RE

In order to show how to solve (1) in general form, let us regard two cases.

2.1 $a(x) = 0$

In case $a(x) = 0$, RE takes particular form

$$\phi_x = b(x)\phi + c(x), \quad (2)$$

[1] See Ex.3

i.e. it is a first-order LODE and its general solution can be expressed in quadratures. As a first step, one has to find a solution $z(x)$ of its homogeneous part[2], i.e.

$$z(x): \quad z_x = b(x)z.$$

In order to find general solution of Eq.(2) let us introduce new variable $\tilde{\phi}(x) = \phi(x)/z(x)$, i.e. $z(x)\tilde{\phi}(x) = \phi(x)$. Then

$$(z(x)\tilde{\phi}(x))_x = b(x)z(x)\tilde{\phi}(x) + c(x), \quad \text{i.e.} \quad z(x)\tilde{\phi}(x)_x = c(x),$$

and it gives us general solution of Eq.(2) in quadratures

$$\phi(x) = z(x)\tilde{\phi}(x) = z(x)(\int \frac{c(x)}{z(x)}dx + \text{const}). \tag{3}$$

This method is called **method of variation of constants** and can be easily generalized for a system of first-order LODEs

$$\vec{y}' = A(x)\vec{y} + \vec{f}(x).$$

Naturally, for the system of n equations we need to know n particular solutions of the corresponding homogeneous system in order to use method of variation of constants. And this is exactly the bottle-neck of the procedure - in distinction with first-order LODEs which are all integrable in quadratures, already second-order LODEs **are not**.

2.2 $a(x) \neq 0$

In this case **one known particular solution of a RE allows to construct its general solution**.

Indeed, suppose that φ_1 is a particular solution of Eq.(1), then

$$c = \varphi_{1,x} - a\varphi_1^2 - b\varphi_1$$

and substitution $\phi = y + \varphi_1$ annihilates free term c yielding to an equation

$$y_x = ay^2 + \tilde{b}y \tag{4}$$

with $\tilde{b} = b + 2a\varphi_1$. After re-writing Eq.(4) as

$$\frac{y_x}{y^2} = a + \frac{\tilde{b}}{y}$$

[2] see Ex.1

and making an obvious change of variables $\phi_1 = 1/y$, we get a particular case of RE

$$\phi_{1,x} + \tilde{b}\phi_1 - a = 0$$

and its general solution is written out explicitly in the previous subsection.

Example 2.1 As an important illustrative example leading to many applications in mathematical physics, let us regard a particular RE in a form

$$y_x + y^2 = x^2 + \alpha. \tag{5}$$

For $\alpha = 1$, particular solution can be taken as $y = x$ and general solution obtained as above yields to

$$y = x + \frac{e^{-x^2}}{\int e^{-x^2} dx + \mathrm{const}},$$

i.e. in case (5) is integrable in quadratures. Indefinite integral $\int e^{-x^2} dx$ though not expressed in elementary functions, plays important role in many areas from probability theory till quantum mechanics.

For arbitrary α, Eq.(5) possess remarkable property, namely, after an elementary fraction-rational transformation

$$\hat{y} = x + \frac{\alpha}{y + x} \tag{6}$$

it takes form

$$\hat{y}_x + \hat{y}^2 = x^2 + \hat{\alpha}, \quad \hat{\alpha} = \alpha + 2,$$

i.e. form of original Eq.(5) did not change while its rhs increased by 2. In particular, after this transformation Eq.(5) with $\alpha = 1$ takes form

$$\hat{y}_x + \hat{y}^2 = x^2 + 3$$

and since $y = x$ is a particular solution of (5), then $\hat{y} = x + 1/x$ is a particular solution of the last equation. It means that for any

$$\alpha = 2k + 1, \quad k = 0, 1, 2, ...$$

general solution of Eq.(5) can be found in quadratures as it was done for the case $\alpha = 1$.

In fact, it means that Eq.(5) is **form-invariant** under the transformations (6). Further we are going to show that general RE possess similar property as well.

2.3 Transformation group

Let us check that general fraction-rational change of variables

$$\hat{\phi} = \frac{\alpha(x)\phi + \beta(x)}{\gamma(x)\phi + \delta(x)} \tag{7}$$

transforms one Riccati equation into the another one similar to Example 2.1. Notice that (7) constitutes group of transformations generated by

$$\frac{1}{\phi}, \quad \alpha(x)\phi, \quad \phi + \beta(x),$$

thus only actions of generators have to be checked:

- $\hat{\phi} = 1/\phi$ transforms (1) into

$$\hat{\phi}_x + c(x)\hat{\phi}^2 + b(x)\hat{\phi} + a(x) = 0,$$

- $\hat{\phi} = \alpha(x)\phi$ transforms (1) into

$$\hat{\phi}_x - \frac{a(x)}{\alpha(x)}\hat{\phi}^2 - [b(x) + (\log\alpha(x))_x]\hat{\phi} - \alpha(x)c(x) = 0,$$

- $\hat{\phi} = \phi + \beta(x)$ transforms (1) into

$$\hat{\phi}_x - a(x)\hat{\phi}^2 + [2\beta(x)a(x) - b(x)]\hat{\phi} - \hat{c} = 0,$$

  where

$$\hat{c} = a(x)\beta^2(x) - b(x)\beta(x) + c(x) + \beta(x)_x.$$

Thus, having **one solution** of a some Riccati equation we can get immediately general solutions of the whole family of REs obtained from the original one under the action of transformation group (7).

It is interesting to notice that for Riccati equation knowing **any three solutions** ϕ_1, ϕ_2, ϕ_3 we can construct all other solutions ϕ using a very simple formula called **cross-ratio**:

$$\frac{\phi - \phi_1}{\phi - \phi_2} = A\frac{\phi_3 - \phi_1}{\phi_3 - \phi_2} \tag{8}$$

with an arbitrary constant A, where choice of A defines a solution. In order to verify this formula let us notice that system of equations

$$\begin{cases} \dot{\phi} = a(x)\phi^2 + b(x)\phi + c(x) \\ \dot{\phi}_1 = a(x)\phi_1^2 + b(x)\phi_1 + c(x) \\ \dot{\phi}_2 = a(x)\phi_2^2 + b(x)\phi_2 + c(x) \\ \dot{\phi}_3 = a(x)\phi_3^2 + b(x)\phi_3 + c(x) \end{cases}$$

is consistent if

$$\begin{bmatrix} \dot{\phi} & \phi^2 & \phi & 1 \\ \dot{\phi}_1 & \phi_1^2 & \phi_1 & 1 \\ \dot{\phi}_2 & \phi_2^2 & \phi_2 & 1 \\ \dot{\phi}_3 & \phi_3^2 & \phi_3 & 1 \end{bmatrix} = 0$$

and direct calculation shows that this condition is equivalent to

$$\frac{d}{dx}\left(\frac{\phi - \phi_1}{\phi - \phi_2} \cdot \frac{\phi_3 - \phi_1}{\phi_3 - \phi_2}\right) = 0. \tag{9}$$

As it was shown, REs are not invariant under the action of (7) while (7) conserves the form of equations but not form of the coefficients. On the other hand, it is possible to construct new differential equations related to a given RE which will be invariant with respect to transformation group (7) (see next section).

At the end of this section we consider a very interesting example [6] showing connection of Eq.(9) with first integrals for generalization of one of Kovalevskii problems [9].

Adler´s example System of equations

$$y_{j,x} + 2y_j^2 = sy_j, \quad s = \sum_{j=1}^{n} y_j, \quad j = 1, 2, ...n \tag{10}$$

was studied by Kovalevskii in case $\boxed{n = 3}$ and it was shown that there exist two quadratic first integrals

$$F_1 = (y_1 - y_2)y_3, \quad F_2 = (y_2 - y_3)y_1$$

and therefore Kovalevskii problem is integrable in quadratures.

In case $\boxed{n \geq 4}$ the use of (9) gives us immediately following some first integrals

$$\frac{y_l - y_i}{y_l - y_j}\frac{y_k - y_i}{y_k - y_j},$$

i.e. Sys.(10) has nontrivial first integrals for arbitrary n.

It is interesting that for this example solution of Sys.(10) is easier to construct without using its first integrals. Indeed, each equation of this system is a Riccati equation if a is regarded as given, substitution $y_i = \phi_{i,x}/2\phi_i$ gives

$$\phi_{i,xx} = s\phi_{i,xx}, \ \phi_{i,xx} = a(x) + c_i, \ s = a_{xx}/a_x$$

and equation for a has form

$$\frac{a_{xx}}{a_x} = \frac{a_x}{2}(\frac{1}{a-c_1} + ... + \frac{1}{a-c_n}).$$

After integration $a_x^2 = \mathrm{const}(a-c_1)...(a-c_n)$, i.e. problem is integrable in quadratures (more precisely, in hyper-elliptic functions).

In fact, one more generalization of Kovalevskii problem can be treated along the same lines - case when function s **is not** sum of y_j but some arbitrary function $s = s(x_1, ..., x_n)$. Then equation on a takes form

$$\frac{a_{xx}}{a_x} = a_x s(\frac{a_x}{a-c_1} + ... + \frac{a_x}{a-c_n})$$

which concludes Adler´s example.

2.4 Singularities of solutions

All the properties of Riccati equations which have been studied till now, are in the frame of local theory of differential equations. We just ignored possible existence of singularities of solutions regarding all its properties locally, in a neighborhood of a point. On the other hand, in order to study analytical properties of solutions, one needs to know character of singularities, behavior of solutions at infinity, etc.

One can distinguish between two main types of singularities - singularities, not depending on initial conditions (they are called **fixed**) and depending on initial conditions (they are called **movable**). Simplest possible singularity is a pole, and that was the reason why first attempt of classification of the ordinary nonlinear differential equations of the first and second order, suggested by Painleve, used this type of singularities as criterium. Namely, list of all equations was written out, having only **poles as movable singularities** (see example of P1 in Chapter 1), and nice analytic properties of their

solutions have been found. It turned out that, in particular, Painleve equations describe self-similar solutions of solitonic equations (i.e. equations in partial derivatives): P2 corresponds to KdV (Korteweg-de Vries equation), P4 corresponds to NLS (nonlinear Schrödinger equation) and so on.

Using cross-ratio formula (8), it is easy to demonstrate for a Riccati equation that **all singularities** of the solution ϕ, with an exception of singularities of particular solutions ϕ_1, ϕ_2, ϕ_3, **are movable poles** described as following:

$$\phi_3 = \frac{1}{1-A}(\phi_2 - A\phi_1)$$

where A is a parameter defining the solution ϕ. Let us construct a solution with poles for Eq.(5) from Example 2.1. We take a solution in a form

$$y = \frac{1}{x+\varepsilon} + a_0 + a_1(x+\varepsilon) + a_2(x+\varepsilon)^2 + ... \tag{11}$$

with indefinite coefficients a_i, substitute it into (5) and make equal terms corresponding to the same power of $(x+\varepsilon)$. The final system of equations takes form

$$\begin{cases} a_0 = 0, \\ 3a_1 - \alpha - \varepsilon^2 = 0, \\ 4a_2 + 2\varepsilon = 0, \\ 5a_3 - 1 + a_1^2 = 0, \\ 6a_4 + 2a_1a_2 = 0, \\ 7a_5 + 2a_1a_3 + a_2^2 = 0 \\ ... \end{cases}$$

and in particular for $\alpha = 3,\ \varepsilon = 0$ the coefficients are

$$a_1 = 1,\ a_2 = a_3 = ... = 0$$

which corresponds to the solution

$$y = x + \frac{1}{x}$$

which was found already in Example 2.1.

This way we have also learned that each pole of solutions have order 1. In general case, it is possible to prove that series (11) converges for arbitrary pair $(\varepsilon,\ \alpha)$ using the connection of RE with the theory of linear equations

(see next section). In particular for fixed complex α, it means that for any point $x_0 = -\varepsilon$ **there exist the only solution of (5)** with a pole in this point.

As to nonlinear first order differential equations (with non-quadratic nonlinearity), they have more complicated singularities. For instance, in a simple example

$$y_x = y^3 + 1$$

if looking for a solution of the form $y = ax^k + ...$ one gets immediately

$$akx^{k-1} + ... = a^3x^{3k} + ... \quad \Rightarrow \ k - 1 = 3k \quad \Rightarrow \ 2k = -1$$

which implies that singularity here is a branch point, not a pole (also see [2]). It make RE also very important while studying degenerations of Painleve transcendents. For instance, (5) describes particular solutions of P4 (for more details see Appendix).

3 Differential equations related to RE

3.1 Linear equations of second order

One of the most spectacular properties of RE is that its theory is in fact **equivalent** to the theory of second order homogeneous LODEs

$$\psi_{xx} = b(x)\psi_x + c(x)\psi \tag{12}$$

because it can easily be shown that these equations can be transformed into Riccati form and *viceversa*. Of course, this statement is only valid if Eq.(1) has non-zero coefficient $a(x)$, $a(x) \neq 0$.

► Indeed, let us regard second-order homogenous LODE (12) and make change of variables

$$\phi = \frac{\psi_x}{\psi}, \quad \text{then} \quad \phi_x = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2},$$

which implies

$$\frac{\psi_{xx}}{\psi} = \phi_x + \frac{\psi_x^2}{\psi^2} = \phi_x + \phi^2$$

and after substituting the results above into initial LODE, it takes form

$$\phi_x = \phi^2 + b(x)\phi + c(x).$$

which is particular case of RE. ∎

◀ On the other hand, let us regard general RE

$$\phi_x = a(x)\phi^2 + b(x)\phi + c(x)$$

and suppose that $a(x)$ is not $\equiv 0$ while condition of $a(x) \equiv 0$ transforms RE into first order linear ODE which can be solved in quadratures analogously to Thomas equation (see Chapter 1). Now, following change of variables

$$\phi = -\frac{\psi_x}{a(x)\psi}$$

transforms RE into

$$-\frac{\psi_{xx}}{a(x)\psi} + \frac{1}{a(x)}\Big(\frac{\psi_x}{\psi}\Big)^2 + \frac{a(x)_x}{a(x)^2}\frac{\psi_x}{\psi} = a(x)\Big(\frac{\psi_x}{a(x)\psi}\Big)^2 - \frac{b(x)}{a(x)}\frac{\psi_x}{\psi} + c(x)$$

and it can finally be reduced to

$$a(x)\psi_{xx} - \big[a(x)_x + a(x)b(x)\big]\psi_x + c(x)a(x)^2\psi = 0$$

which is second order homogeneous LODE. ∎

Now, analog of the result of Section 2.2 for second order equations can be proved.

Proposition 3.1 *Using one solution of a second order homogeneous LODE, we can construct general solution as well.*

▶ First of all, let us prove that without loss of generality we can put $b(x) = 0$ in $\psi_{xx} + b(x)\psi_x + c(x)\psi = 0$. Indeed, change of variables

$$\psi(x) = e^{-\frac{1}{2}\int b(x)dx}\hat{\psi}(x) \quad \Rightarrow \quad \psi_x = (\hat{\psi}_x - \frac{1}{2}b\hat{\psi})e^{-\frac{1}{2}\int b(x)dx}$$

and finally

$$\hat{\psi}_{xx} + \hat{c}\hat{\psi} = 0, \quad \hat{c} = c - \frac{1}{4}b^2 - \frac{1}{2}b_x. \tag{13}$$

Now, if we know one particular solution $\hat{\psi}_1$ of Eq.(13), then it follows from the considerations above that RE

$$\phi_x + \phi^2 + \hat{c}(x) = 0$$

has a solution $\phi_1 = \hat{\psi}_{1,x}/\hat{\psi}_1$. The change of variables $\hat{\phi} = \phi - \phi_1$ annihilates the coefficient $\hat{c}(x)$:

$$(\hat{\phi} + \phi_1)' + \left((\hat{\phi} + \phi_1)^2\right) + \hat{c}(x) = 0 \quad \Rightarrow \quad \hat{\phi}_x + \hat{\phi}^2 + 2\phi_1\hat{\phi} = 0 \quad \Rightarrow$$

$$\left(\frac{1}{\hat{\phi}}\right)_x = 1 + 2\phi_1\frac{1}{\hat{\phi}}, \tag{14}$$

i.e. we reduced our RE to the particular case Eq.(2) which is integrable in quadratures. Particular solution $z = 1/\hat{\phi}$ of homogeneous part of Eq.(14) can be found from

$$z_x = 2z\frac{\hat{\psi}_{1,x}}{\hat{\psi}_1} \quad \text{as} \quad z = \hat{\psi}_1^2$$

and Eq.(3) yields to

$$\hat{\psi}_2(x) = \hat{\psi}_1 \int \frac{dx}{\hat{\psi}_1^2(x)}. \tag{15}$$

Obviously, two solutions $\hat{\psi}_1$ and $\hat{\psi}_2$ are linearly independent since Wronskian $< \hat{\psi}_1, \hat{\psi}_2 >$ is non-vanishing[3]:

$$< \hat{\psi}_1, \hat{\psi}_2 >:= \begin{vmatrix} \hat{\psi}_1 & \hat{\psi}_2 \\ \hat{\psi}_1' & \hat{\psi}_2' \end{vmatrix} = \hat{\psi}_1\hat{\psi}_2' - \hat{\psi}_2\hat{\psi}_1' = 1 \neq 0.$$

Thus their linear combination gives general solution of Eq.(12).∎

Proposition 3.2 Wronskian $< \psi_1, \psi_2 >$ is constant **iff** ψ_1 and ψ_2 are solutions of

$$\psi_{xx} = c(x)\psi. \tag{16}$$

▶ Indeed, if ψ_1 and ψ_2 are solutions, then

$$(\psi_1\psi_2' - \psi_2\psi_1')' = \psi_1\psi_2'' - \psi_2\psi_1'' = c(x)(\psi_1\psi_2 - \psi_1\psi_2) = 0 \quad \Rightarrow$$

$$\Rightarrow \quad \psi_1\psi_2' - \psi_2\psi_1' = \text{const}\,.$$

∎

◀ if Wronskian of two functions ψ_1 and ψ_2 is a constant,

$$\psi_1\psi_2' - \psi_2\psi_1' = \text{const} \quad \Rightarrow \quad \psi_1\psi_2'' - \psi_2\psi_1'' = 0$$

[3]see Ex.2

$$\Rightarrow \quad \frac{\psi_2''}{\psi_2} = \frac{{\psi_1}''}{\psi_1}.$$

$\blacksquare$

Conservation of the Wronskian is one of the most important characteristics of second order differential equations and will be used further for construction of modified Schwarzian equation.

To illustrate procedure described in Proposition 3.1, let us take **Hermite equation**

$$\emptyset_{xx} - 2x\emptyset_x + 2\lambda\emptyset = 0. \tag{17}$$

Change of variables $z = \emptyset_x/\emptyset$ yields to

$$z_x = \frac{\emptyset_{xx}}{\emptyset} - z^2, \quad z_x + z^2 - 2xz + 2\lambda = 0$$

and with $y = z - x$ we get finally

$$y_x + y^2 = x^2 - 2\lambda - 1,$$

i.e. we got the equation studied in Example 2.1 with $\alpha = -2\lambda - 1$. It means that all solutions of Hermite equation with positive integer λ, $\lambda = n$, $n \in \mathbb{N}$ can easily be found while for negative integer λ one needs change of variables inverse to (6):

$$y = -x + \frac{\gamma}{\hat{y} - x}, \quad (\hat{y} - x)(y + x) = \gamma, \quad \gamma = \hat{\alpha} - 1, \quad \hat{\alpha} = \alpha + 2.$$

It gives us **Hermite polynomials**

$$\begin{cases} \lambda = 0, \quad y = -x, \quad \emptyset = 1 \\ \lambda = 1, \quad y = -x + \frac{1}{x}, \quad \emptyset = 2x \\ \lambda = 2, \quad y = -x + \frac{4x}{2x^2 - 1)} \quad \emptyset = 4x^2 - 2 \\ \dots\dots\dots\dots \\ \lambda = n, \quad y = -x + \frac{\emptyset_x}{\emptyset}, \quad \emptyset = H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \\ \dots\dots\dots \end{cases}$$

as solutions.

Notice that the same change of variables

$$\phi = \frac{\psi_x}{\psi} \tag{18}$$

which linearized original RE, was also used for linearization of Thomas equation and Burgers equation in Chapter 1. This change of variables is

called **log-derivative** of function ψ or $D_x \log(\psi)$ and plays important role in many different aspects of integrability theory, for instance, when solving factorization problem.

Theorem 3.3 Linear ordinary differential operator L of order n could be factorized with factor of first order , i.e. $L = M \circ (\partial_x - a)$ for some operator M, **iff**

$$a = \frac{\psi_x}{\psi}, \quad \text{where} \quad \psi \in \mathrm{Ker}(L). \tag{19}$$

► $L = M \circ (\partial_x - a),\ a = \psi_x/\psi$ implies $(\partial_x - a)\psi = 0$, i.e. $\psi \in \mathrm{Ker}(L)$. ∎

◄ Suppose that $\psi_1 = 1$ is an element of the $\mathrm{Ker}(L)$, i.e. $\psi_1 \in \mathrm{Ker}(L)$. It leads to $a = 0$ and operator L has zero free term and is therefore divisible by ∂_x.

If constant function $\psi_1 = 1$ does not belong to the kernel of initial operator, following change of variables

$$\hat{\psi} = \frac{\psi}{\psi_1}$$

lead us to a new operator

$$\hat{L} = f^{-1} L \circ f \tag{20}$$

which has a constant as a particular solution $\hat{\psi}_1$ for $f = \psi_1$. ∎

Remark. Operators L and $\hat{L}$ given by (20), are called **equivalent operators** and their properties will be studied in detailed in the next Chapter.

Notice that Theorem 3.3 is analogous to the Bezout´s theorem on divisibility criterium of a polynomial: A polynomial $P(z) = 0$ is divisible on the linear factor, $P(z) = P_1(z)(z - a)$, iff a is a root of a given polynomial, i.e. $P(a) = 0$. Thus, in fact this classical theorem constructs one to one correspondence between factorizability and solvability of $L(\psi) = 0$.

The factorization of differential operators is in itself a very interesting problem which we are going to discuss in details in Chapter 3. Here we will

only regard one very simple example - LODO with constant coefficients

$$L(\psi) := \frac{d^n\psi}{dx^n} + a_1\frac{d^{n-1}\psi}{dx^{n-1}} + \ldots + a_n\psi = 0.$$

In this case each root λ_i of a **characteristic polynomial**

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$$

generates a corresponding first order factor with

$$\lambda_i = \frac{\psi_x}{\psi}$$

ant it yields to

$$\psi_x = \lambda_i\psi \quad \Rightarrow \quad \psi = c_i e^{\lambda_i x}$$

and finally

$$L = \frac{d^n}{dx^n} + a_1\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_n = (\frac{d}{dx} - \lambda_1)\cdots(\frac{d}{dx} - \lambda_n).$$

This formula allows us to construct general solution for $L(\psi) = 0$, i.e. for $\psi \in \mathrm{Ker}(L)$, of the form

$$\psi = \sum c_i e^{\lambda_i x}$$

in the case of all distinct roots of characteristic polynomial.

In case of double roots λ_k with multiplicity m_k it can be shown that

$$\psi = \sum P_k(x)e^{\lambda_k x} \tag{21}$$

where degree of a polynomial $P_k(x)$ depends on the multiplicity of a root, $degP_k(x) \leq m_k - 1$ (cf. Ex.4)

3.2 Schwarzian equation

Let us regard again second-order LODE

$$\psi_{xx} + b(x)\psi_x + c(x)\psi = 0 \tag{22}$$

and suppose we have two solutions ψ_1, ψ_2 of (22). Let us introduce new function $\varphi = \psi_1/\psi_2$, then

$$\varphi_x = \frac{\psi_{1x}\psi_2 - \psi_{2x}\psi_1}{\psi_2^2}, \quad \varphi_{xx} = b(x)\frac{\psi_{1x}\psi_2 - \psi_{2x}\psi_1}{\psi_2^2} + 2\frac{\psi_{1x}\psi_2 - \psi_{2x}\psi_1}{\psi_2^3}\psi_{2x},$$

which yields

$$\frac{\varphi_{xx}}{\varphi_x} = -b(x) - 2\frac{\psi_{2x}}{\psi_2}$$

and substituting $\phi = \frac{\psi_{2x}}{\psi_2} = (\log \psi_2)_x$ into (1) related to (22) we get finally

$$\frac{3}{4}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 - \frac{1}{2}\frac{\varphi_{xxx}}{\varphi_x} = c(x). \tag{23}$$

Left hand of (23) is called **Schwarz derivative** or just **Schwarzian** and is invariant in respect to transformation group (7) with constant coefficients $\alpha, \beta, \gamma, \delta$:

$$\hat{\varphi} = \frac{\alpha\varphi + \beta}{\gamma\varphi + \delta}.$$

It is sufficient to check only two cases:

$$\hat{\varphi} = \frac{1}{\varphi} \text{ and } \hat{\varphi} = \alpha\varphi + \beta.$$

which can be done directly.

This equation plays major role in the theory of conform transformations of polygons [7].

3.3 Modified Schwarzian equation

Notice that substitution $\varphi = \psi_1/\psi_2$ allowed us to get invariant form of the initial Eq.(22). Another substitution, namely, $\varphi = \psi_1\psi_2$, leads to similar equation which differs from classical Schwarzian equation (23) only by a constant term and this is the reason why we call it **modified Schwarzian equation**. This form of Schwarzian equation turns out to be useful for a construction of approximate solutions of Riccati equations with parameter (see next section). In order to construct modified Schwarzian equation, we need following Lemma.

Lemma 3.4 Let $\psi_1,\ \psi_2$ are two linear independent solutions of

$$\psi_{xx} = c(x)\psi. \tag{24}$$

Then functions

$$\psi_1^2, \quad \psi_2^2, \quad \psi_1\psi_2$$

constitute a basis in the solution space of the following third order equation:

$$\varphi_{xxx} = 4c(x)\varphi_x + 2c_x(x)\varphi. \tag{25}$$

▶ Using notations

$$\varphi_1 = \psi_1^2,\ \ \varphi_2 = \psi_2^2,\ \ \varphi_1 = \psi_1\psi_2,$$

we can compute Wronskian $\mathcal{W}$ of these three functions

$$\mathcal{W} =< \varphi_1, \varphi_2, \varphi_3 >= (\psi_1\psi_{2,x} - \psi_2\psi_{1,x})^3 =< \psi_1, \psi_2 >^3$$

and use Proposition 3.2 to demonstrate that

$$\mathcal{W} = \text{const} \neq 0,$$

i.e. functions φ_i are linearly independent.

After introducing notations

$$\mathcal{V} =< \psi_1, \psi_2 > \quad \text{and} \quad f_j = \frac{\psi_{j,x}}{\psi_j}$$

it is easy to obtain

$$\frac{\mathcal{V}}{\varphi_3} = f_2 - f_1, \quad \frac{\varphi_{3,x}}{\varphi_3} = f_2 + f_1$$

which yields to

$$f_1 = \frac{\varphi_{3,x} - \mathcal{V}}{2\varphi_3}, \quad f_2 = \frac{\varphi_{3,x} + \mathcal{V}}{2\varphi_3}. \tag{26}$$

Substitution of these f_j into

$$f_{j,x} + f_j^2 = c(x)$$

gives

$$4c(x)\varphi^2 + \varphi_x^2 - 2\varphi\varphi_{xx} = \mathcal{V}^2. \tag{27}$$

with $\varphi = \varphi_3$ and differentiation of Eq.(27) with respect to x gives Eq.(25) and it easy to see that **equations (27) and (25) are equivalent**.

Analogous reasoning shows that $\varphi_1,\ \varphi_2$ are also solutions of Eq.(25). ∎

Equation (25) as well as its equivalent form (27) will be used further for construction of approximate solutions of REs, they also define solitonic hierarchies for KdV and NLS. It will be more convenient to use (27) in slightly

different form.

Let us rewrite (27) as

$$4c(x) + \frac{\varphi_x^2}{\varphi^2} - \frac{2\varphi_{xx}}{\varphi} = \frac{\mathcal{V}^2}{\varphi^2}$$

and introduce notation $a = 1/\varphi$, then

$$c(x) = \frac{3}{4}\frac{a_x^2}{a^2} - \frac{1}{2}\frac{a_{xx}}{a} + \mathcal{V}^2 a^2 \tag{28}$$

and compare this equation with Schwarzian equation

$$c(x) = \frac{3}{4}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 - \frac{1}{2}\frac{\varphi_{xxx}}{\varphi_x}$$

one can see immediately why Eq.(28) is called **modified Schwarzian equation**.

Notice that after the substitution $a = e^{2b}$, rhs of modified Schwarzian equation, i.e. **modified Schwarzian derivative, Dmod**, takes a very simple form

$$Dmod(a) := \frac{3}{4}\frac{a_x^2}{a^2} - \frac{1}{2}\frac{a_{xx}}{a} = b_{xx} + b_x^2$$

which is in a sense similar to $D_x \log$. Indeed, for $\psi = e^{\varphi}$,

$$D_x \log(\psi) = \frac{\psi_x}{\psi} = \varphi_x = e^{-\varphi}\frac{d}{dx}e^{\varphi},$$

while

$$Dmod(e^{2\varphi}) = e^{-\varphi}\frac{d^2}{dx^2}e^{\varphi}.$$

At the end of this section let us stress the following basic fact: we have shown that from some very logical point of view, first order nonlinear Riccati equation, second order linear equation and third order nonlinear Schwarzian equation **are equivalent**! It gives us freedom to choose the form of equation which is most adequate for specific problem to be solved.

4 Asymptotic solutions

In our previous sections we have studied Riccati equation and its modifications as classical ordinary differential equations, with one independent variable. But many important applications of second order differential equations consist some additional parameter λ, for instance one of the most significant equations of one-dimensional quantum mechanics takes one of two forms

$$\psi_{xx} = (\lambda + u)\psi \tag{29}$$
$$\psi_{xx} = (\lambda^2 + u_1\lambda + u_2)\psi \tag{30}$$

where Eq.(29) is called Schrödinger equation and Eq.(30) can be considered as modified Dirac equation in quantum mechanics while in applications to solitonic hierarchies it is called Zakharov-Shabat equation. Notice that Schrödinger equation with $u(x) = x^2$ equivalent to Eq.(17). Coefficients of these two equations have special names - u, u_1, u_2 are called **potentials** due to many physical applications and λ is called **spectral parameter** because of following reason. Schrödinger equation, being rewritten as

$$L(\psi) = \lambda\psi, \quad L(\psi) = \psi_{xx} - u\psi$$

becomes obviously an equation for eigenfunctions of operator L (with appropriate boundary conditions, of course). This operator is called *Schrödinger operator*.

For our convenience we name the whole coefficient before ψ as **generalized potential** allowing it sometimes to be a polynomial in λ of **any finite degree**. Coming back to Eq.(24), the generalized potential is just the function $c(x)$.

Now, with the equation having a parameter, problem of its integrability became, of course, more complicated and different approaches can be used to solve it. If we are interested in a solution for all possible values of a parameter λ, asymptotic solution presented by a formal series can always be obtained (section 4.1) while for some specific exact solutions can be constructed (sections 4.2, 4.3) in a case of truncated series. It becomes possible while existence of a parameter gives us one more degree of freedom to play with. Cf. with Example 4.2 where exact solution has been obtained also as a series and its convergence resulted from the main theorem of the theory of differential equations on solvability of Cauchy problem. On the other hand, this solution is valid only for some restricted set of parameter´ values, namely for integer odd α.

4.1 RE with a parameter λ

Let us show first that the RE with a parameter λ corresponding to Eq.(30), namely

$$f_x + f^2 = \lambda^2 + u_1\lambda + u_2, \quad \text{with} \quad f = D_x \log(\psi), \tag{31}$$

has a solution being represented as a formal series.

Lemma 4.1 Eq.(31) has a solution

$$f = \lambda + f_0 + \frac{f_1}{\lambda} + ... \tag{32}$$

where coefficients f_j are differential polynomials in u_1 and u_2.

► After direct substituting the series (32) into the equation for f and making equal corresponding coefficients in front of the same powers of λ, we get

$$\begin{cases} 2f_0 = u_1 \\ 2f_1 + f_{0,x} + f_0^2 = u_2 \\ 2f_2 + f_{1,x} + 2f_0 f_1 = 0 \\ 2f_3 + f_{2,x} + 2f_0 f_2 + f_1^2 = 0 \\ ..... \end{cases}$$

and therefore, coefficients of (32) are differential polynomials of potentials u_1 and u_2. ∎

Notice that taking a series

$$g = -\lambda + g_0 + \frac{g_1}{\lambda} + \frac{g_2}{\lambda^2} + \frac{g_3}{\lambda^3} \quad ... \tag{33}$$

as a form of solution , we will get a different system of equations on its coefficients g_i:

$$\begin{cases} -2g_0 = u_1 \\ -2g_1 + g_{0,x} + g_0^2 = u_2 \\ -2g_2 + g_{1,x} + 2g_0 g_1 = 0 \\ -2g_3 + g_{2,x} + 2g_0 g_2 + g_1^2 = 0 \\ ..... \end{cases}$$

Solution of Eq.(31) constructed in Lemma 4.1 yields to the solution of original Zakharov-Shabat equation (30) of the form

$$\psi_1(x,\lambda) = e^{\int f(x,\lambda)dx} = e^{\lambda x}(\eta_0(x) + \frac{\eta_1(x)}{\lambda} + \frac{\eta_2(x)}{\lambda^2} + \frac{\eta_3(x)}{\lambda^3} + ...) \tag{34}$$

and analogously, the second solution is

$$\psi_2(x,\lambda) = e^{\int g(x,\lambda)dx} = e^{-\lambda x}(\xi_0(x) + \frac{\xi_1(x)}{\lambda} + \frac{\xi_2(x)}{\lambda^2} + \frac{\xi_3(x)}{\lambda^3} + ...) \tag{35}$$

In fact, it can be proven that Wronskian $< \psi_1, \psi_2 >$ is a power series on λ (see Ex.8) with constant coefficients. Notice that existence of these two solutions is not enough to construct general solution of initial Eq.(30) because linear combination of these formal series is not defined, also convergence problem has to be considered. On the other hand, existence of Wronskian in a convenient form allows us to construct family of potentials giving convergent series for (34) and (35). We demonstrate it at the more simple example, namely Schrödinger equation (29).

Let us regard Schrödinger equation (29), its RE has form

$$f_x + f^2 = \lambda + u, \quad \text{with} \quad f = D_x \log(\psi), \tag{36}$$

and it can be regarded as particular case of (30), i.e. the series for f yields to

$$f = k + f_0 + \frac{f_1}{k} + ..., \quad \lambda = k^2, \tag{37}$$

and $g(x,k) = f(x,-k)$. We see that in case of (29) there exists a simple way to calculate function g knowing function f and it allows us to construct two solutions of Schrödinger equation (29):

$$\psi_1(x,k) = e^{\int f(x,k)dx} = e^{kx}(1 + \frac{\zeta_1(x)}{k} + \frac{\zeta_2(x)}{k^2} + \frac{\zeta_3(x)}{k^3} + ...) \tag{38}$$

and

$$\psi_2(x,k) = \psi_1(x,-k).$$

Substitution of say ψ_1 into (29) gives a recurrent relation between coefficients ζ_i:

$$\zeta_{j+1,x} = \frac{1}{2}(u\zeta_j - \zeta_{j,xx}), \quad \zeta_0 = 1. \tag{39}$$

In particular,

$$u = 2\zeta_{1,x} \tag{40}$$

which means that in order to compute potential u it is enough to know only **one coefficient** ζ_1 of the formal series! Below we demonstrate how this recurrent relation helps us to define potentials corresponding to a given solution.

Example 4.2 Let us regard truncated series corresponding to the solutions of (29)

$$\psi_1 = e^{kx}(1+\frac{\zeta_1}{k}), \quad \psi_2 = e^{-kx}(1-\frac{\zeta_1}{k}),$$

then due to (39)

$$u = 2\zeta_{1,x}, \quad \zeta_{1,xx} = 2\zeta_{1,x}\zeta_1$$

and Wronskian $\mathcal{W}$ of these two functions has form

$$\mathcal{W} =< \psi_1, \psi_2 >= -2k + \frac{1}{k}(\zeta_1^2 - \zeta_{1,x}). \tag{41}$$

Notice that

$$(\zeta_1^2 - \zeta_{1,x})_x = 0$$

and it means that $\mathcal{W}$ does not depend on x, $\mathcal{W} = \mathcal{W}(k)$. Introducing notation k_1 for a zero of the Wronskian, $\mathcal{W}(k_1) = 0$, it is easy to see that

$$\zeta_1^2 - \zeta_{1,x} = k_1^2$$

which implies that ψ_1 and ψ_2 are solutions of (29) with

$$\zeta_1 = k_1 - \frac{2k_1}{1+e^{-2k_1(x-x_0)}} = -k_1 \tanh k_1(x-x_0)$$

and potential

$$u = -2\frac{(2k_1)^2}{(e^{k_1(x-x_0)} + e^{-k_1(x-x_0)})^2} = \frac{-2k_1^2}{\cosh^2(k_1(x-x_0))}, \tag{42}$$

where x_0 is a constant of integration.

It is important to understand here that general solution of Schrödinger equation (29) can be now found as a linear combination of ψ_1 and ψ_2 for all values of a parameter $\lambda = k^2$ **with exception** of two special cases: $k = 0$ and $k = k_1$ which implies functions ψ_1 and ψ_2 are **linearly dependent** in these points.

Fig.1 (...)

At the Fig. 1 graph of potential u is shown and it is easy to see that magnitude of the potential in the point of extremum is defined by zeros of the Wronskian $\mathcal{W}$. At the end of this Chapter it will be shown that this potential represents a solitonic solution of stationary KdV equation, i.e. **solution of a Riccati equation generates solitons!**

4.2 Soliton-like potentials

It this section we regard only Schrödinger equation (29) and demonstrate that generalization of the Example 4.2 allows us to describe a very important special class of potentials having solutions in a form of truncated series.

Definition 4.3 Smooth real-valued function $u(x)$ such that

$$u(x) \to 0 \quad \text{for} \quad x \to \pm\infty,$$

is called **transparent potential** if there exist solutions of Schrödinger equation (29) in a form of truncated series with potential $u(x)$.

Another name for a transparent potential is **soliton-like** or **solitonic** potential due to many reasons. The simplest of them is just its form which is a bell-like one and "wave" of this form was called a soliton by [10] and this notion became one of the most important in the modern nonlinear physics, in particular while many nonlinear equations have solitonic solutions.

Notice that truncated series ψ_1 and ψ_2 can be regarded as polynomials in k of some degree N multiplied by some exponent (in Example 4.2 we had $N = 1$). In particular, it means that Wronskian $\mathcal{W} =< \psi_1, \psi_2 >$ is odd function, $\mathcal{W}(-k) = -\mathcal{W}(k)$, vanishing at $k = 0$ and also it is a **polynomial** in k of degree $2N + 1$:

$$\mathcal{W}(k) = -2k \prod_1^N (k^2 - k_j^2). \tag{43}$$

As in Example 4.2, functions ψ_1 and ψ_2 are linearly dependent at the points k_j, i.e.

$$\psi_1(x, k_j) = A_j \psi_2(x, k_j), \quad j = 1, 2, ..., N$$

with some constant proportionality coefficients A_j.

Theorem 4.4 Suppose we have two sets of real positive numbers

$$\{k_j\}, \quad \{B_j\}, \quad j = 1, 2, ..., N, \quad k_j, B_j > 0, \quad k_j, B_j \in \mathbb{R}$$

such that numbers k_j are ordered in following way

$$k_1 > k_2 > ... > k_N > 0$$

and B_j are arbitrary. Let functions $\psi_1(x,k), \psi_2(x,k)$ have form

$$\psi_1(x,k) = e^{kx}(k^N + a_1 k^{N-1} + ... + a_N), \quad \psi_2(x,k) = (-1)^N \psi_1(x,-k)$$

with indefinite real coefficients $a_j, j = 1, 2, ..., N$.

Then there exist unique set of numbers $\{a_j\}$ such that two functions $\psi_1(x,k_j), \ \psi_2(x,k_j)$ satisfy system of equations

$$\psi_2(x,k_j) = (-1)^{j+1} B_j \psi_1(x,k_j) \tag{44}$$

and Wronskian of these two functions has form

$$\mathcal{W}(k) = -2k \prod_1^N (k^2 - k_j^2). \tag{45}$$

► Our first step is to construct a_j. It is easy to see that (44) is equivalent to the following system of equations on $\{a_j\}$ (for simplicity the system is written out for a case $N = 3$)

$$\begin{cases} k_1^2 a_1 + E_1 k_1 a_2 + a_3 + k_1^3 E_1 = 0 \\ k_2^2 E_2 a_1 + k_2 a_2 + E_2 a_3 + k_2^3 = 0 \\ k_3^2 a_1 + E_3 k_3 a_2 + a_3 + k_3^3 E_3 = 0 \end{cases} \tag{46}$$

where following notations has been used:

$$E_j = \frac{e^{\tau_j} - e^{-\tau_j}}{e^{\tau_j} + e^{-\tau_j}} = \tanh \tau_j, \quad \tau_j = k_j x + \beta_j, \quad B_j = e^{2\beta_j}.$$

(To show this, it is enough to write out explicitly ψ_1 and ψ_2 in roots of polynomial and regard two cases: N is odd and N is even. For instance, if $k = k_1$ and N is odd, we get

$$\psi_1(x,k) = e^{kx}(k^N + a_1 k^{N-1} + ... + a_N) = ...\psi_2(x,k) = (-1)^N \psi_1(x,-k)...$$

)

Obviously, $0 \leq E_j < 1$ and for a case $E_j = 1, \forall j = 1, 2, 3$ the system (46) takes form

$$\begin{cases} k_1^2 a_1 + k_1 a_2 + a_3 + k_1^3 = 0 \\ k_2^2 a_1 + k_2 a_2 + a_3 + k_2^3 = 0 \\ k_3^2 a_1 + k_3 a_2 + a_3 + k_3^3 = 0 \end{cases} \tag{47}$$

(...). Thus, it was shown that determinant of Sys.(46) is non-zero, i.e. all a_j are uniquely defined and functions $\psi_1(x, k_j)$, $\psi_2(x, k_j)$ are polynomials.

In order to compute the Wronskian $\mathcal{W}$ of these two functions, notice first that $\mathcal{W}$ is a polynomial with leading term $-2k^{2N+1}$. Condition of proportionality (44) for functions $\psi_1(x, k_j)$, $\psi_2(x, k_j)$ provides that k_j are zeros of the Wronskian and that $\mathcal{W}$ is an odd function on k, i.e. (45) is proven. ∎

In order to illustrate how to use this theorem for construction of exact solutions with transparent potentials let us address two cases: $N = 1$ and $N = 2$.

Example 4.5 In case $N = 1$ explicit form of functions

$$\psi_1 = e^{kx}(k + a_1), \quad \psi_2 = e^{-kx}(k - a_1)$$

allows us to find a_1 immediately:

$$a_1 = -k_1 E_1 = -k_1 \tanh y_1 = -k_1 \tanh (k_1 x + \beta_1)$$

which coincides with formula for a solution of the same equation obtained in Example 4.2

$$\zeta_1 = -k_1 \tanh k_1 (x - x_0)$$

for $x_0 = \beta_1 / k_1$. As to potential u, it can be computed as above using recurrent relation which keeps true for all N.

Notice that using this approach we have found solution of Schrödinger equation by **pure algebraic means** while in Example 4.2 we had to solve Riccati equation in order to compute coefficients of the corresponding truncated series.

The system (46) for case $N = 2$ takes form

$$\begin{cases} k_1 a_1 + E_1 a_2 + k_1^2 E_1 = 0 \\ E_2 k_2 a_1 + a_2 + k_2^2 = 0 \end{cases}$$

which yields to

$$
\begin{aligned}
a_1 &= \frac{(k_2^2 - k_1^2)E_1}{k_1 - k_2 E_1 E_2} \\
&= D_x \log\left((k_2 - k_1)\cosh(\tau_1 + \tau_2) + (k_2 + k_1)\cosh(\tau_1 - \tau_2)\right) \qquad (48)
\end{aligned}
$$

and corresponding potential $u = 2a_{1,x}$ has explicit form

$$
u = 2D_x^2 \log\left((k_2 - k_1)\cosh(\tau_1 + \tau_2) + (k_2 + k_1)\cosh(\tau_1 - \tau_2)\right) \qquad (49)
$$

where

$$
x \to \pm\infty \qquad \Rightarrow \; a_1 \to \pm(k_1 + k_2),
$$

i.e. u is a smooth function such that

$$
u \to 0 \quad \text{for} \quad x \to \pm\infty.
$$

Formulae (48) and (49) have been generalized for the case of arbitrary N by Hirota whose work gave a rise to a huge amount of papers dealing with construction of soliton-like solutions for **nonlinear differential equations** because some simple trick allows to add new variables in these formulae (see [8] and bibliography herein). For instance, if we take

$$
\tau_j = k_j x + \beta_j = k_j x + k_j^2 y + k_j^3 t
$$

then formula (49) gives particular solutions of Kadomtzev-Petviashvili (KP) equation

$$
(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0 \qquad (50)
$$

which is important model equation in the theory of surface waves.

4.3 Finite-gap potentials

In previous section it was shown how to construct integrable cases of Schrödinger equation with soliton-like potentials vanishing at infinity. Obvious - but not at all a trivial - next step is to generalize these results for construction of integrable cases for Schrödinger equation with **periodic potentials**. In the pioneering work [11] the finite-gap potentials were introduced and described in terms of their spectral properties but deep discussion of spectral theory lays beyond the scope of this book (for exhaustive review see, for instance,

[12]). The bottleneck of present theory of finite-gap potentials is following: spectral properties formulated by Novikov´s school provide only almost periodic potentials but do not guarantee periodic ones in all the cases.

We are going to present here some simple introductory results about finite-gap potentials and discuss a couple of examples. For this purpose, most of the technique demonstrated in the previous section can be used though as an auxiliary equation we will use not Riccati equation but its equivalent form, modified Schwarzian (28).

Generalization of Lemma 4.1 for the case of arbitrary finite polynomial $c(x)$ can be formulated as follows.

Lemma 4.6 Equation for modified Schwarzian

$$\frac{3}{4}\frac{h_x^2}{h^2} - \frac{1}{2}\frac{h_{xx}}{h} + \lambda^m h^2 = U(x,\lambda) := \lambda^m + u_1\lambda^{m-1} + ... + u_m \tag{51}$$

with any polynomial generalized potential $U(x,\lambda)$ has unique asymptotic solution represented by formal Laurant series such that:

$$h(x,\lambda) = 1 + \sum_{k=1}^{\infty} \lambda^{-k} h_k(x) \tag{52}$$

where coefficients h_j are differential polynomials in all $u_1, ..., u_m$.

► The proof can be carried out directly along the same lines as for Lemma 4.1. ∎

Direct corollary of this lemma is following: coefficients of the formal solution are explicit functions of potential $U(x,\lambda)$. In particular, for $m = 1$ which corresponds to Schrödinger equation (29) with generalized polynomial potential $\lambda + u$ we have

$$h_1 = \frac{1}{2}u, \quad 2h_2 = \frac{1}{2}h_{1,xx} - h_1^2, ... \tag{53}$$

Definition 4.7 Generalized potential

$$U(x,\lambda) = \lambda^m + u_1\lambda^{m-1} + ... + u_m$$

of an equation

$$\psi_{xx} = U(x,\lambda)\psi$$

is called N**-phase potential** if Eq. (25),

$$\varphi_{xxx} = 4U(x,\lambda)\varphi_x + 2U_x(x)\varphi,$$

has a solution φ which is a polynomial in λ of degree N:

$$\varphi(x,\lambda) = \lambda^N + \varphi_1(x)\lambda^{N-1} + ... + \varphi_N(x) = \prod_{j=1}^{N}(\lambda - \gamma_j(x)). \qquad (54)$$

Roots $\gamma_j(x)$ of the solution $\varphi(x,\lambda)$ are called **root variables**.

In particular case of Schrödinger equation this potential is also called **finite-gap potential**. As it follows from [12], original "spectral" definition of a finite-gap potential is equivalent to our Def. 4.7 which is more convenient due to its applicability not only for Schrödinger equation but also for arbitrary equation of the second order.

There exists direct connection between the statement of Lemma 4.6 and the notion of finite-gap potential. One can check directly that if function $h(x,\lambda)$ is solution of Eq.(51), then function $\varphi(x,\lambda) = 1/h(x,\lambda)$ is solution of Eq.(25) and can be written as formal series

$$\varphi(x,\lambda) = 1 + \sum_{k=1}^{\infty} \lambda^{-k}\varphi_k(x) \qquad (55)$$

with coefficients which are explicit functions of generalized potential $U(x,\lambda)$. In case when series (55) becomes a finite sum, we get finite-gap potential.

Example 4.8 Let a solution $\varphi(x,\lambda) = \lambda - \gamma(x)$ is a polynomial of first degree and potential is also linear, i.e. $m = 1,\ N = 1$. Then after integrating the equation from definition above, we get

$$4(\lambda + u)\varphi^2 + \varphi_x^2 - 2\varphi\varphi_{xx} = c(\lambda) \qquad (56)$$

with some constant of integration $c(\lambda)$ and left part of (56) is a polynomial in λ of degree 3,

$$C(\lambda) = 4\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 4(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3), \qquad (57)$$

where λ_i are all roots of the polynomial $C(\lambda)$. Eq.(56) is identity on λ and therefore without loss of generality we write further $C(\lambda)$ for both sides of

it. This identity has to keep true for all values of λ, in particular, also for $\lambda = \gamma(x)$ which gives

$$\gamma_x^2 = C(\gamma) = 4(\gamma - \lambda_1)(\gamma - \lambda_2)(\gamma - \lambda_3). \tag{58}$$

Now instead of solving Eq.(56), we have to solve Eq.(58) which is integrable in elliptic functions.

If we are interested in real solutions without singularities, we have to think about initial data for Eq.(58). For instance, supposing that all λ_j are real, without loss of generality

$$\lambda_1 > \lambda_2 > \lambda_3,$$

and for initial data (x_0, γ_0) satisfying

$$\forall (x_0, \gamma_0): \ \lambda_3 < \gamma_0 < \lambda_2,$$

Eq.(58) has real smooth periodic solution expressed in elliptic functions

$$u = 2\gamma - \lambda_1 - \lambda_2 - \lambda_3 \tag{59}$$

It is our finite-gap potential (1-phase potential) and its period can be computed explicitly as

$$T = \int_{\lambda_3}^{\lambda_2} \frac{d\lambda}{\sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)}}.$$

We have regarded in Example 4.8 particular case $mN = 1$. Notice that in general case of $mN > 1$ following the same reasoning, after integration we get polynomial $C(\lambda)$ of degree $2N + m$

$$4U(x, \lambda)\varphi^2 + \varphi_x^2 - 2\varphi\varphi_{xx} = C(\lambda) := 4\lambda^{2N+m} + ... \tag{60}$$

and correspondingly a system of $2N + m - 1$ equations on N functions, i.e. the system will be over-determined. On the other hand, choice of $\lambda = \gamma_j$ makes it possible to get a closed subsystem of N equations for N functions as above:

$$\gamma_{j,x}^2 = C(\gamma_j) / \prod_{j \neq k} (\gamma_j - \gamma_k)^2. \tag{61}$$

Following Lemma shows that this over-determined system of equations has unique solution which is defined by Sys.(61).

Dubrovin´s Lemma Let system of differential equations (61) on root variables γ_j defined by (54) is given with

$$C(\lambda) = 4\lambda^{2N+m} + ...,$$

then following keeps true:
1. $C(\lambda)|_{\lambda=\gamma_j} = \varphi_x^2(x,\lambda)|_{\lambda=\gamma_j},\ j = 1, ..., N,$
2. expression

$$\varphi^{-1}(2\varphi_{xx} + \frac{C(\lambda) - \varphi_x^2}{\varphi})$$

is a polynomial in λ of degree m and leading coefficient 1.

◀ Let us notice that

$$\big(\prod(\lambda - \gamma_k)\big)_x|_{\lambda=\gamma_j} = \big(-\gamma_{1,x}\prod_{j=2}^{N}(\lambda-\gamma_j) - .... - \gamma_{N,x}\prod_{j=1}^{N-1}(\lambda-\gamma_j)\big)|_{\lambda=\gamma_j}$$

$$= -\gamma_{j,x}\prod_{j\neq k}(\gamma_j - \gamma_k)$$

which implies

$$\varphi_x|_{\lambda=\gamma} = -\gamma_{j,x}\prod_{j\neq k}(\gamma_j - \gamma_k) \ \Rightarrow\ C(\lambda)|_{\lambda=\gamma_j} = \varphi_x^2(x,\lambda)|_{\lambda=\gamma_j},$$

i.e. first statement of the lemma is proven.

(...). ▮

Below it will be shown that also **transparent potentials themselves** can be computed algebraically. (...)

5 Summary

In this Chapter, using Riccati equation as our main example, we tried to demonstrate at least some of the ideas and notions introduced in Chapter 1 - integrability in quadratures, conservation laws, etc. Regarding transformation group and singularities of solutions for RE, we constructed some equivalent forms of Riccati equation. We also compared three different approaches to the solutions of Riccati equation and its equivalent forms. The classical form of RE allowed us to construct easily asymptotic solutions represented by formal series. Linear equation of the second order turned out to be

more convenient to describe finite-gap potentials for exact solitonic solutions which would be a much more complicated task for a RE itself while generalization of soliton-like potentials to finite-gap potentials demanded modified Schwarzian equation.

In our next Chapter we will show that modified Schwarzian equation also plays important role in the construction of a differential operator commuting with a given one while existence of commuting operators allows us to obtain examples of hierarchies for solitonic equations using Lemma 4.6. In particular, for $m = 1$ coefficients $h_k(x)$ of Eq.(52) describe a set of conservation laws for **Korteweg-de Vries equation** (**KdV**)

$$u_t + 6uu_x + u_{xxx} = 0. \tag{62}$$

6 Exercises for Chapter 2

1. Prove that general solution of $z' = a(x)z$ has a form

$$z(x) = e^{\int a(x)dx}.$$

2. Deduce formula (15) regarding $< \hat{\psi}_1, \hat{\psi}_2 >= 1$ as a linear first order equation on $\hat{\psi}_2$.

3. Prove that for $L = \frac{d^m}{dx^m}$ its kernel consists of all polynomials of degree $\leq m - 1$.

4. Let functions A_1 and A_2 are two solutions of (25). Prove that the Wronskian $< A_1,\ A_2 >= A_1A_2' - A_2A_1'$ is solution as well.

5. Let the function A satisfies (27). Prove that the functions

$$f_{\pm} = \frac{1}{2}D\log A \pm \frac{\sqrt{z}}{A}$$

satisfies the Riccati equations (1).

6. Proof that

$$\frac{3}{4}\frac{a_x^2}{a^2} - \frac{1}{2}\frac{a_{xx}}{a} = k^2 \ \Leftrightarrow \ a = (\varepsilon_1 e^{kx} + \varepsilon_2 e^{-kx})^{-2}.$$

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